

Mathematical Proofs of the Binomial Theorem

Question

Use mathematical induction to prove the binomial theorem. (The binomial theorem is $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$.)

Answer

Base Case:

$$(x + y)^0 = \sum_{k=0}^0 \binom{0}{k} x^k y^{0-k} = \binom{0}{0} x^0 y^0 = 1 \cdot 1 = 1$$

$$(x + y)^1 = \sum_{k=0}^1 \binom{1}{k} x^k y^{1-k} = \binom{1}{0} x^0 y^1 + \binom{1}{1} x^1 y^0 = 1 \cdot y + 1 \cdot x = x + y$$

Inductive Step:

$$(x + y)^{n+1} = (x + y)^n (x + y) = \left(\sum_{k=0}^n \binom{n}{k} x^k y^{n-k} \right) (x + y)$$

$$= \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} x + \sum_{k=0}^n \binom{n}{k} x^k y^{n-k} y$$

Reindexing (1):

$$= \sum_{k=0}^n \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^n \binom{n}{k} x^k y^{n-k+1}$$

$$= \sum_{k=0}^n \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{k-1} y^{n-k+1}$$

Reindexing (2):

$$= \sum_{k=0}^n \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^n \binom{n}{k} x^k y^{n-k+1}$$

$$= \sum_{k=0}^n \left(\binom{n}{k} + \binom{n}{k} \right) x^k y^{n-k+1}$$

Reindexing (3):

$$= \sum_{k=0}^n \binom{n+1}{k} x^k y^{n-k+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

Additional Information